FINAL: COMMUTATIVE ALGEBRA I

Date: **15th November 2016**
The Total points is **114** and the maximum you can score is **100** points.

A **ring** would mean a **commutative ring with identity**.

(1) (3+3+3+3+3+3+3+3=24 points) Let R be a noetherian ring, I be a proper ideal in R. State **true or false**. (Proof not required).
   (a) If I is a free R-module then I is a principal ideal.
   (b) If S is a multiplicative subset of R then $S^{-1}R$ is a flat R-module.
   (c) If R is a local ring then every finitely generated projective R-module is free.
   (d) If I is a prime ideal then I^n is primary for all $n \geq 0$.
   (e) If I is a primary ideal then $I = P^n$ for some prime ideal P in R.
   (f) If I is a primary ideal of R then its radical $\sqrt{I}$ is a prime ideal.
   (g) R is artinian then $\mathrm{Spec}(R)$ is finite.
   (h) $\mathrm{Spec}(R)$ is finite then R is artinian.

(2) (10 points) Let R be a ring and S a multiplicative subset. Show that the contraction and extension of ideals between R and $S^{-1}R$ induces a bijection between $\{P \in \mathrm{Spec}(R) : P \cap S = \emptyset\}$ and $\mathrm{Spec}(S^{-1}R)$.

(3) (5+15=20 points) Define integral extension. Let $A \subset B$ be integral extension of finite dimensional rings. Show that dimension of A and B are same.

(4) (5+15=20 points) State Noether normalization theorem. Let k be a field of characteristic 0 and R be an integral domain which is also a finitely generated k-algebra. Show that the integral closure of R in its fraction field is a noetherian ring.

(5) (10+5+5=20 points) Let k be a field, $R = k[X, Y, Z]$ be the polynomial ring and $I = (X^2, XY, YZ, ZX)$. Compute the minimal prime ideals of I. Show that (X, Y, Z) is radical of a primary ideal which appear in a minimal primary decomposition of I. Is the minimal primary decomposition unique?

(6) (5+15=20 points) Define valuation ring. Let R be a discrete valuation ring and m be its maximal ideal. Let $\hat{R}$ be the m-adic completion of R. Show that $\hat{R}$ is a discrete valuation ring.